The shared reward dilemma

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Abstract

One of the most direct human mechanisms of promoting cooperation is rewarding it. We study the effect of sharing a reward among cooperators in the most stringent form of social dilemma, namely the prisoner's dilemma (PD). Specifically, for a group of players that collect payoffs by playing a pairwise PD game with their partners, we consider an external entity that distributes a fixed reward equally among all cooperators. Thus, individuals confront a new dilemma: on the one hand, they may be inclined to choose the shared reward despite the possibility of being exploited by defectors; on the other hand, if too many players do that, cooperators will obtain a poor reward and defectors will outperform them. By appropriately tuning the amount to be shared a vast variety of scenarios arises, including the traditional ones in the study of cooperation as well as more complex situations where unexpected behavior can occur. We provide a complete classification of the equilibria of the n-player game as well as of its evolutionary dynamics.
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Keywords: Reward; Social dilemma; Prisoner's dilemma; n-Player game; Cooperation; Evolutionary dynamics; Nash equilibria

1. Introduction

Selfish behavior seems to be one of the consequences of evolutionary dynamics. Genes, organisms, generic entities acting in their own benefit do better in a struggle for reproductive (understood in a wide sense) success and are selected in the long term. In spite of this general trend, we find in every evolutionary context (be it biological, sociological, economic, etc.) many instances in which cooperative behaviors are evolutionarily successful. The explanation of this puzzle has developed into an active line of research, and providing a complete answer to it is one of the big open problems of XXI century (Pennisi, 2005). Many mechanisms have been identified as respon-

sible for these cooperative associations. Among them we find kinship (Hamilton, 1964a, b), reciprocity (Axelrod and Hamilton, 1981), reputation gain (Nowak and Sigmund, 1998), and others (Axelrod, 1984; Nowak, 2006). One of the most interesting mechanisms of this kind that has been identified is altruistic punishment and rewarding (Sigmund et al., 2001) or voluntary participation (Hauert et al., 2007). Through this mechanism social groups that are engaged in social dilemmas, such as the one represented by the public goods game, can overcome the well-known tragedy of the commons (Hardin, 1968).

The rewarding mechanisms just mentioned are of the bottom-up type, i.e., they arise at the individual level and lead to cooperation at the group level. However, in ecological and social contexts, there are several levels of organization which make possible top-down approaches. For instance, parents, educators, governments, and other institutions promote prosocial behavior by rewarding individuals in different manners (prizes, incentives, tax

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deductions, etc.). In biological or ecological contexts, some species reward symbionts that cooperate at the required level by providing them with more resources (see Kiers et al., 2003 and references therein). Companies also use similar mechanisms in their own benefit to induce customers to supply useful information about consumption habits or social networks (Iribarren and Moro, 2007). Finally, another instance of top-down rewarding can be found in team formation of animal societies (Anderson and Franks, 2001), e.g. in cooperative hunting (Packer and Ruttan, 1988).

Top-down rewarding mechanisms can be generically implemented in two different ways. The simplest one is to provide a fixed benefit to every cooperator. In terms of game theory, this is tantamount to shifting the payoff matrix by a constant added to entries related to cooperation. Thus, for instance, if one starts off with a prisoner's dilemma (PD) to model the baseline social behavior, introducing such a reward transforms the dilemma into another one, either snowdrift (Maynard-Smith and Price, 1973; Sugden, 1986) or stag hunt (Skyrms, 2003), or even suppresses completely the dilemma, changing it into a harmony game (Licht, 1999). A second, more subtle mechanism is to distribute a fixed amount between all cooperators in the population. In this case, the original PD becomes a new dilemma, because there is a clear incentive to cooperate but if there are too many cooperators the incentive disappears and hence defecting pays. This is reminiscent of the minority game paradigm (Moro, 2004) and, in fact, it may be seen as an alternative form of describing situations in which being in the minority (understood in a lax sense) is the best option. We will refer to this situation as the *shared reward dilemma*.

In this work we study the shared reward dilemma by considering an interaction group of n individuals. In order to understand it in the most stringent form of social dilemma, interaction among individuals follows the PD (see Doebeli and Hauert, 2005 for a review). Thus, we introduce a game in which payoffs can be obtained from two sources: first, all players collect payoffs by playing a n-player generalization of the PD game with their partners (Hauert and Szabó, 2003), and second, players who have chosen to cooperate share an extra payoff coming from a pool. In the next section we analyze in detail the n-player game. Situations in which multiple interior equilibria occur are completely determined, as well as the parametric settings in which equilibria increase, decrease, or jump discontinuously with the reward. In Section 3 we analyze the evolutionary stability of the equilibria discussed in Section 2 and provide the different asymptotic scenarios of cooperation according to the replicator dynamics. Section 4 summarizes our conclusions and presents some future prospects. Appendix A contains the main mathematical results on which the discussions of previous sections rest: a theorem and a corollary that provide closed formulae for the symmetric Nash equilibria in terms of the reward for finite and large number of players, respectively. To complete our analysis, we present in Appendix B a theorem which characterizes all asymmetric Nash equilibria in pure strategies of the game.

2. The shared reward dilemma

Consider an assembly of n players, each of whom can choose one out of two actions: cooperate (C) or defect (D) with the rest of the $n - 1$ players in a one-shot game (i.e., all player's actions are simultaneously performed). Players collect payoffs according to a PD game from every one of the $n - 1$ opponents. In addition, players who have chosen to cooperate obtain an extra payoff coming from a fixed reward ρ, provided by an external source, that is evenly distributed among all cooperators.

To provide the strategic form of this game we introduce some notation. Let k be the number of cooperators in the group. Payoffs of pairwise interactions are denoted by the standard parameters of the PD game: a defector that exploits a cooperator obtains the temptation T, but when she faces up another defector she receives the punishment P; instead, the payoff for a cooperator meeting another cooperator is the reward R (not to be confused with ρ, the reward to be shared that we propose in this work), but obtains the sucker's payoff S when she confronts a defector. For the game to be a PD, the payoffs must be ordered according to $T > R > P > S$. Since the game is symmetric, in the sense that the payoff to a particular player is independent of her label and only depends on her actions, the total payoff of an arbitrary player is given by

$$U = \begin{cases} (k-1)R + (n-k)S + \dfrac{\rho}{k} & \text{if she cooperates,} \\ kT + (n-1-k)P & \text{if she defects.} \end{cases} \quad (1)$$

The remaining of this section is devoted to study the Nash equilibria of this game.

Let us begin with the symmetric Nash equilibria in pure strategies, which can be easily obtained from (1). Full cooperation is an equilibrium if no player increases her payoff by defecting unilaterally, that is, if and only if $T(n-1) \leqslant (n-1)R + \rho/n$. Similarly, full defection is an equilibrium if no player increases her payoff by cooperating unilaterally, i.e., if and only if $(n-1)S + \rho \leqslant (n-1)P$. The former constraint on ρ suggests a normalization of the shared reward, namely

$$\delta = \frac{\rho}{n(n-1)(T-R)}, \quad (2)$$

which will henceforth be referred to as *scaled reward*. With this parameter, the condition for full cooperation to be a Nash equilibrium is simply $\delta \geqslant 1$. As for the second constraint, if we introduce a new parameter, the *defection ratio*

$$\zeta = \frac{T-R}{P-S}, \quad (3)$$

the condition for full defection to be a Nash equilibrium is $\delta \leqslant 1/n\zeta$. All the analysis of the game can be performed solely in terms of these two parameters instead of the five parameters that originally define the game. As we have shown, the scaled reward is the ratio between the actual reward and the reward needed for full cooperation to be a Nash equilibrium; as for the defection ratio, it compares, in a pairwise interaction, the excess of payoff a defector gets over a cooperator when both confront a cooperator, with that when both face up a defector.

Note that both full defection and full cooperation will coexist if and only if $1 \leqslant \delta \leqslant 1/n\zeta$. Clearly, no reward meets this condition unless $\zeta \leqslant 1/n$. Thus we see that, by increasing the reward, the symmetric Nash equilibrium in pure strategies changes from full defection to full cooperation, and in between these two extremes there may be either coexistence or absence of both equilibria, depending on whether ζ is smaller or larger than $1/n$, respectively.

The space of symmetric mixed strategies Nash equilibria consists of all $0 \leqslant q \leqslant 1$ such that a player cooperates with probability q and defects with probability $1-q$. The expected total payoffs of an arbitrary cooperator and of an arbitrary defector when the rest of the players play an equilibrium q, are given by

$$f_C(q) = \mathbb{E}[U|\text{she cooperates}] = (n-1)qR + (n-1)(1-q)S + \rho\mu_{n-1}(q), \qquad (4)$$

$$f_D(q) = \mathbb{E}[U|\text{she defects}] = (n-1)qT + (n-1)(1-q)P, \qquad (5)$$

where $\mu_m(q) = \mathbb{E}[(S_m + 1)^{-1}]$, S_m being a binomial random variable which is the sum of m i.i.d. Bernoulli's random variables with mean q. As has been observed by Chao and Strawderman (1972), $\mu_m(q)$ has the expression

$$\mu_m(q) = \begin{cases} 1 & \text{for } q = 0, \\ \dfrac{1 - (1-q)^{m+1}}{(m+1)q} & \text{for } 0 < q \leqslant 1. \end{cases} \qquad (6)$$

Symmetric Nash equilibria in completely mixed strategies can be computed by solving $f_C(q) = f_D(q)$. To do that, it is convenient to distinguish when there are more than two players and when there are just two players involved. The latter case is particularly simple because it reproduces the major binary games used in the study of cooperation. The payoff matrix (Gintis, 2000) of this binary game can be easily obtained from (1) by setting $n = 2$, and it is shown in Table 1. Thus, depending on ρ, the game becomes a:

(i) prisoner's dilemma, if $T > R + \rho/2$ and $P > S + \rho$;
(ii) snowdrift, if $T > R + \rho/2$ and $P < S + \rho$;
(iii) stag hunt, if $T < R + \rho/2$ and $P > S + \rho$;
(iv) harmony, if $T < R + \rho/2$ and $P < S + \rho$.

The Nash equilibria of these games are well known. Thus, the snowdrift game has two asymmetric Nash equilibria in pure strategies, $\{(C,D),(D,C)\}$, while the stag-hunt game

has two symmetric Nash equilibria, $\{(C,C),(D,D)\}$. Both games have a unique Nash equilibrium in mixed strategies $q \in (0,1)$. Otherwise, the PD and the harmony game have just one Nash equilibrium (both players defecting and both cooperating, respectively).

In terms of δ and ζ, the above conditions (i)–(iv) can be rephrased as

(i′) prisoner's dilemma if $\delta < \min(1, 1/2\zeta)$;
(ii′) snowdrift if $1/2\zeta < \delta < 1$;
(iii′) stag hunt if $1 < \delta < 1/2\zeta$;
(iv′) harmony if $\delta > \max(1, 1/2\zeta)$.

In general, our results permit to characterize the changes in the structure of equilibria by varying δ and fixing ζ. Therefore, we can study the effect of rising the reward. In order to illustrate our approach, consider once more the binary game. Upon increasing δ the game changes from PD to harmony. For $\zeta = 1/2$ this change occurs directly when δ crosses at 1, but depending on whether $\zeta > 1/2$ or $\zeta < 1/2$, the change occurs via snowdrift or via Stag-hunt, respectively.

Taking $n = 2$ in (4) and (5) (hence $\mu_1(q) = 1 - q/2$) and solving $f_C(q) = f_D(q)$ we obtain a unique Nash equilibrium in mixed strategies $0 < q < 1$ given by

$$q = \frac{1 - 2\delta\zeta}{1 - (1+\delta)\zeta}. \qquad (7)$$

If $\zeta > 1/2$ (respectively $\zeta < 1/2$) q is a continuous increasing (respectively, decreasing) function of δ. Fig. 1 illustrates these two scenarios as well as the parametric conditions for the existence and coexistence of equilibria in pure strategies. When δ lies in between 1 and $1/2\zeta$, there is uncertainty as to the strategy that players will choose: for

Table 1
Payoff matrix for the binary case of the shared reward dilemma

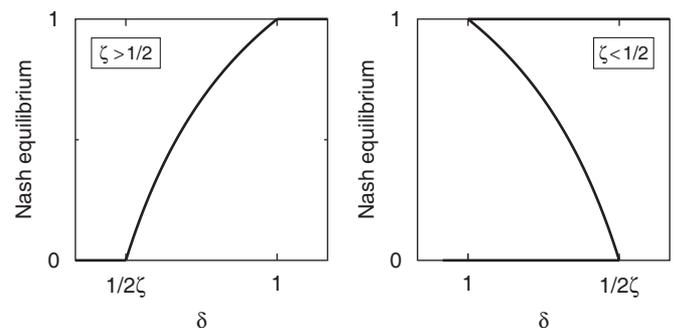|	C	D
C	$R + \rho/2$	$S + \rho$
D	T	P

Fig. 1. Symmetric Nash equilibria of the binary game as a function of the scaled reward δ for the two types of possible behavior, $\zeta > 1/2$ (left) and $\zeta < 1/2$ (right).

$\zeta > 1/2$, because no symmetric Nash equilibrium in pure strategies exists when $1/2\zeta < \delta < 1$; for $\zeta < 1/2$, because there is coexistence of both full cooperation and full defection in the range $1 < \delta < 1/2\zeta$. In the former case the mixed strategies Nash equilibrium that fills the gap has the expected behavior: the probability of cooperating increases with the reward; however, in the latter case the behavior of this Nash equilibrium is counterintuitive, as the probability of cooperating decreases with the reward. This phenomenon can be explained in the framework of evolutionary dynamics, where the binary game models pairwise interactions between individuals of a large population. In this context, it is well known that, under the replicator dynamics, the equilibrium in mixed strategies of the stag hunt game is unstable and separates the basins of attraction of the two equilibria in pure strategies (full defection and full cooperation). We will come back to this issue in Section 3 in a more general setting, where we study in detail the replicator dynamics by considering interactions in groups of n individuals.

Let us now analyze the case $n \geqslant 3$. Notice that $\mu_{n-1}(q)$ defined in (6) is now a nonlinear function of q and thus there can be more than one solution of $f_C(q) = f_D(q)$. However, as such solutions are obtained as the intersection points of a straight line with a strictly convex function, there can be up to two equilibria in the open interval $(0,1)$. As is proven in Theorem 1 of Appendix A, the number of equilibria depends only on the values of δ and ζ. Moreover, the changes on the structure of equilibria when δ increases correspond to three possible scenarios, determined by $\zeta < 1/n$, $1/n \leqslant \zeta < 1/2$ and $\zeta \geqslant 1/2$. (Notice that for $n = 2$ the middle case is empty, and the other two cases correspond to those discussed above.) Fig. 2 depicts the typical structure of equilibria for these three cases.

For the case $\zeta \geqslant 1/2$, Theorem 1 shows that there exists a unique symmetric Nash equilibrium which is a continuous increasing function of δ. It is strictly increasing within $[1/n\zeta, 1]$ from full defection at $\delta = 1/n\zeta$ to full cooperation at $\delta = 1$, and constant outside the interval. However, when $\zeta < 1/2$ we have two nontrivial, different scenarios. One feature common to both of them is the existence of a range of rewards, namely $\max\{1, 1/n\zeta\} < \delta < \delta_c$, for which two

symmetric equilibria in mixed strategies coexist. One of these equilibria increases and the other decreases when the reward increases within this range. At the critical value δ_c these equilibria collapse and a further increase in δ yields a discontinuous jump from a Nash equilibrium with $q < 1$ to full cooperation. An upper bound for δ_c is provided in Theorem 1. The fundamental difference between the cases $\zeta < 1/n$ and $1/n < \zeta < 1/2$ arises in the region $\min\{1, 1/n\zeta\} < \delta < \max\{1, 1/n\zeta\}$, where there exists a unique equilibrium $0 < q < 1$: for $1/n < \zeta < 1/2$ we see that q increases with δ, while for $\zeta < 1/n$, we see that q decreases with δ, exhibiting the same counterintuitive behavior reported for the binary case.

A case of particular importance is $\zeta = 1$, because it reproduces the cost/benefit parametrization of the PD game, by letting $T = b$, $R = b - c$, $P = 0$ and $S = -c$, with $b > c > 0$. For this popular framework, suitable for biological applications, our result shows that the equilibrium of the shared reward dilemma only depends on the fixed amount ρ to be shared by the cooperators and on the cost c of cooperation, but it is independent of the benefit b. An analogous result is observed in a spatial evolutionary version of the shared reward dilemma (Jiménez et al., 2007).

When the number of players $n \to \infty$, we provide a simplified asymptotic version of Theorem 1, in Corollary 1 of Appendix A. As in this limit the threshold $1/n\zeta \to 0$, the third of the three cases shown in Fig. 2 disappears. Notice that in order to get $0 < \delta < \infty$ in the $n \to \infty$ limit, we have to scale the reward with the number of interactions in the game, $n(n-1)$. The reason is that the payoffs collected per player from their pairwise interactions, in the first step of the game, are $O(n)$, therefore the reward per player must be of the same order to produce an effect. This makes $\rho = O(n^2)$. In that case, the shapes of the first two cases in Fig. 2 are preserved, with a shift of the threshold $1/n\zeta$ to 0 (full defection is an equilibrium if and only if $\rho = o(n^2)$). The critical value of the scaled reward, δ_c, at which the equilibrium jumps discontinuously from a value $q < 1$ to full cooperation when $\zeta < 1/2$, can be exactly computed in the asymptotic case $n \to \infty$. As it is proved in Corollary 1, $\delta_c = 1/4\zeta(1 - \zeta)$.
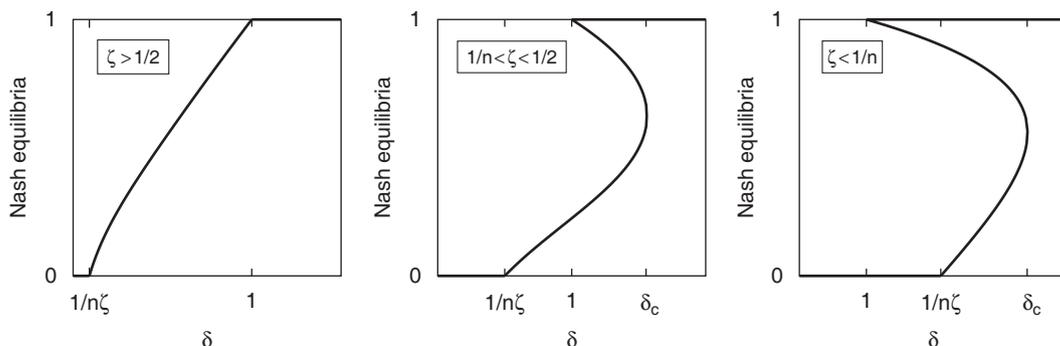

Fig. 2. Symmetric Nash equilibria of the n-player game ($n \geqslant 3$) as a function of δ for the three types of possible behavior, $\zeta > 1/2$ (left), $1/n < \zeta < 1/2$ (middle), and $\zeta < 1/n$ (right).

The limit case $\zeta \to +\infty$ (equivalent to $P \to S^+$) has also received special attention in the analysis of PD games on complex networks (Nowak and May, 1992; Nowak and Sigmund, 2000; Eguíluz et al., 2005). Our results show (c.f. Eq. (A.1)) that a well-defined mixed Nash equilibrium exists for $0 < \delta < 1$ which monotonically increases with δ from 0 to 1, reaching full cooperation for $\delta \geqslant 1$. In the $n \to \infty$ limit, using Corollary 1, we can obtain an estimate for the equilibrium when $P \to S^+$, namely the smallest value between $\sqrt{\delta}$ and 1.

Asymmetric Nash equilibria in pure strategies, in which part of the players in the group cooperate and the rest defect, can also be found for this game. For an interval of rewards starting at $1/n\zeta$ (the maximum reward for which full defection is a Nash equilibrium) there exist asymmetric equilibria with k cooperators and $n - k$ defectors. The value of k increases stepwise, starting from $k = 1$, at reward values $1/n\zeta = \delta_1 < \delta_2 < \ldots$ (see Eq. (B.1)), with equilibria with k and $k + 1$ cooperators coexisting precisely and only at the separating values δ_k. For instance, upon increasing δ above $1/n\zeta$, the full defection equilibrium is replaced by one with a single cooperator and $n - 1$ defectors. In turn, this is the only Nash equilibrium in pure strategies up δ_2, where it is replaced by another equilibrium with two cooperators and $n - 2$ defectors. The maximum number of cooperators in asymmetric equilibria is $n - 1$ if $\zeta \geqslant 1/2$, or else the largest integer $k \leqslant (n-1)/2(1-\zeta)$ if $\zeta < 1/2$. In order to complete the analysis of the static game, a full characterization of these equilibria is given by Theorem 2 of Appendix B. There is a particular aspect of them which we would like to call attention upon: the fraction of cooperators in the asymmetric Nash equilibria approaches either the unique or the lowest mixed strategies Nash equilibrium $0 < q < 1$ in the limit $n \to \infty$. As we will see in Section 3, for the study of the replicator dynamics based on the shared reward dilemma, only the knowledge of symmetric Nash equilibria is necessary.

3. Evolutionary dynamics

In population dynamics, the evolution of cooperation can be modeled in several ways. According to the replicator dynamics (Hofbauer and Sigmund, 1998), the dynamics in infinitely large populations is described by

$$\frac{dx}{dt} = x(1-x)[f_C(x) - f_D(x)], \tag{8}$$

$x(t)$ being the fraction of cooperators at time t and $f_C(x)$ and $f_D(x)$ the average fitness (which is the evolutionary counterpart of the concept of payoff) of cooperators and defectors in the population, respectively. In this paper we consider the approach presented by Hauert et al. (2006) to study replicator dynamics based on interaction groups of individuals. The standard setup to obtain the replicator equation is to assume a large population of individuals who randomly select partners to play a two-person game. In this alternative approach, players select groups of $n - 1$ individuals and play an n-person game instead. This is an appropriate approach to study the evolutionary behavior of populations interacting through Public Goods games (Hauert et al., 2006), and it is also suitable to study the evolutionary behavior of the shared reward dilemma.

If the population is well mixed, the number of cooperators at time t in an interaction group of n individuals is a binomial random variable with mean $nx(t)$. Therefore, the average fitnesses at time t are given by formulae (4) and (5) with $q = x(t)$. Inserting these formulae in (8) we model the evolution of cooperation when a reward ρ is available for each interaction group.

It is clear that $x = 0$ and 1 are always fixed points of the replicator equation (8), but there will be further fixed points at the solutions of $f_C(x^*) = f_D(x^*)$ in the open interval (0,1). All of them are the symmetric Nash equilibria discussed in previous section. By the *folk theorem* of evolutionary game theory (Cressman, 2003), the asymptotic stability of these fixed points will depend on the sign of $f_C(x) - f_D(x)$. For example, if it is always positive, $x = 0$ is unstable whereas $x = 1$ is stable, and if it is always negative it is the other way around. The situation is different if $f_C(x) - f_D(x)$ changes sign in the interval (0,1). By Theorem 1 (see Appendix A), we can determine how many roots (none, one or two) has $f_C(x) - f_D(x)$ in the open interval $(0, 1)$. On the other hand, since $f_C(0) - f_D(0) = n(n-1)(T-R)(\delta - 1/n\zeta)$, then $x = 0$ is stable if $\delta < 1/n\zeta$ and it is unstable otherwise. Thus, we will find the following stability patterns, depending on the number of roots of (A.1) in the interval (0,1):

(I) if $\delta < 1/n\zeta$ (in this case there is either none or just one root),
  (a) if there are no roots, $x = 0$ is a stable and $x = 1$ an unstable fixed point;
  (b) if there is one root $0 < x_1 < 1$, then $x = 0$ is a stable, x_1 an unstable and $x = 1$ a stable fixed point, with x_1 separating the basins of attraction of $x = 0$ and $x = 1$;
(II) if $\delta > 1/n\zeta$,
  (a) if there are no roots, $x = 0$ is an unstable and $x = 1$ a stable fixed point;
  (b) if there is one root $0 < x_1 < 1$, then $x = 0$ is an unstable, x_1 is a stable and $x = 1$ an unstable fixed point;
  (c) if there are two roots $0 < x_1 < x_2 < 1$, then $x = 0$ is an unstable, x_1 stable, x_2 an unstable and $x = 1$ a stable fixed point, and x_2 separates the basins of attraction of x_1 and $x = 1$.

All these situations are illustrated in Fig. 3. Obviously the structure of fixed points of the replicator equation is the same as that of the symmetric Nash equilibria described in the previous section. The only difference is that now $x = 0$ and 1 are always fixed points. What is really new is the stability patterns induced by the dynamics. These patterns are shown in Fig. 3 through flux lines which indicate the
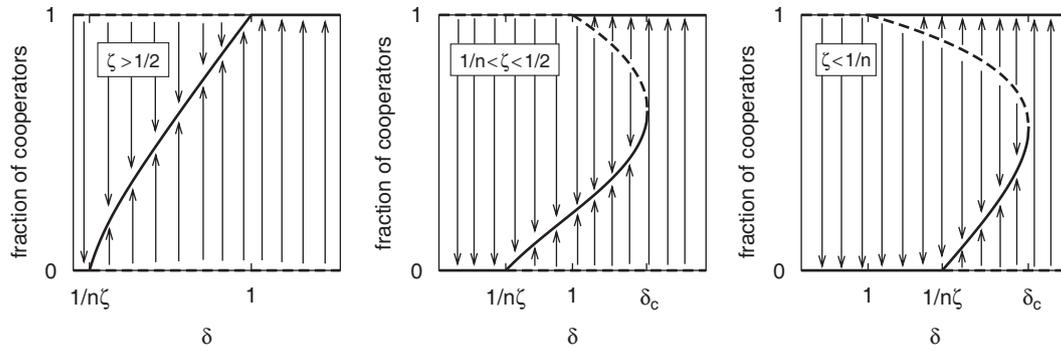
Fig. 3. Equilibria of the replicator equation (8). Solid lines represent the asymptotically stable fixed points, while dashed lines represent the unstable ones.
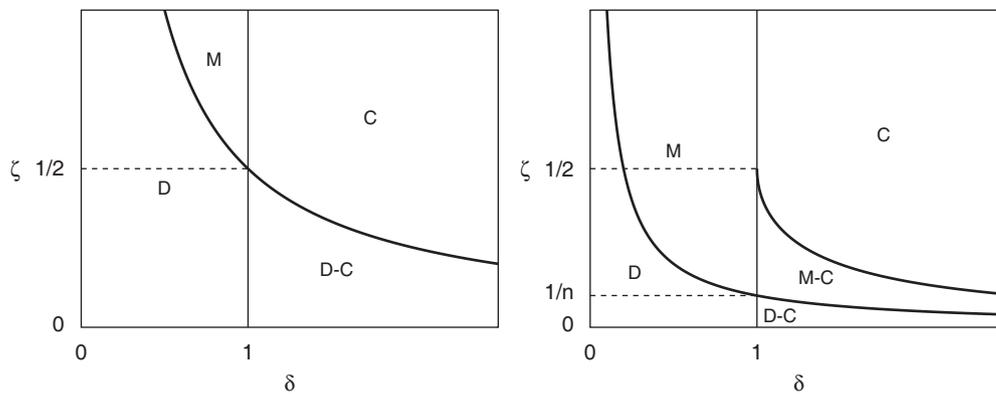

Fig. 4. Diagrams sketching the different dynamical regimes for $n = 2$ (left) and $n \geqslant 3$ (right) in terms of the two parameters ζ (defection ratio) and δ (scaled reward). Symbols stand for full defection (D), full cooperation (C), co-existence of defectors and cooperators in a mixed equilibrium (M), and existence of two stable equilibria, either full defection/full cooperation (D–C) or mixed equilibrium/full cooperation (M–C), each of which is reached depending on the fraction of cooperators in the initial population. The curve marking the upper bound for the D and D–C regions in both figures is given by $\zeta = 1/n\delta$. As $n \to \infty$ this curve moves toward the lower-left corner, thus shrinking these two regions, which disappear in the strict limit. The other curve of the right figure corresponds to the value of δ at which the two mixed equilibria which are found for $\zeta < 1/2$ coalesce (δ_c; see text).

direction in which the dynamics approaches the stable equilibria. It is worth noticing that for the two cases with $\zeta < 1/2$ (middle and right panels of Fig. 3) there is a critical value of the reward, δ_c, at which, starting from a zero fraction of cooperators, the asymptotic cooperation level jumps discontinuously from a value $q < 1$ to full cooperation. In both of them there is also a region of δ in which, depending on the initial fraction of cooperators, the outcome may be full cooperation or a smaller fraction of cooperators. This smaller fraction outcome may even be 0 in the case in which $\zeta < 1/n$. An important consequence is that, $x = 0$ being unstable for any $\delta > 1/n\zeta$, for a suitable reward, a single mutant in an interaction group of defectors will spread cooperation in the population.

To complete our analysis, we summarize the different dynamical regimes that can be obtained, by varying δ and ζ, in Fig. 4. These diagrams illustrate the transitions between the different evolutionary outcomes: full defection, coexistence of cooperators and defectors, bi-stability—where full defection or full cooperation can be

reached, depending on the initial population—full cooperation, and—only for $n \geqslant 3$ players—bi-stability between a mixed population and full cooperation.

4. Conclusions

In this paper we have studied the effect of rewarding cooperation in a strict social dilemma through the distribution of a fixed amount among all cooperative individuals. By adding this payment to the standard payoffs of the prisoner's dilemma, cooperators and defectors in an interaction group confront a dilemma: on the one hand, individuals may be inclined to choose for shared reward despite the possibility of being exploited by defectors; on the other hand, if too many players do that, cooperators will obtain a poor reward and defectors will outperform them. In the simplest case with only two players, we recover the traditional binary games for the study of cooperation where the social dilemma is relaxed: stag hunt and snowdrift.

Although intuition suggests that in this game there should be a threshold value of the reward above which cooperation increases monotonically up to reaching saturation, the game exhibits more complex situations. The equilibrium structure has been characterized for the static game as well as for an evolutionary version of the game based on the replicator dynamics. For a wide range of parameters, scenarios with multiple interior equilibrium points are obtained, featuring critical values of the reward at which cooperation jumps discontinuously. Also, counterintuitive behavior where cooperation decreases as the reward increases may be observed. On the other hand, the replicator dynamics provides additional stability criteria for these equilibria. In the light of the stability patterns that arise, counterintuitive equilibria in the static game, exhibiting a decrease of cooperation upon increasing reward, turn out to be unstable equilibria of the dynamics separating basins of attraction of other stable equilibria. As a consequence, a most relevant conclusion is that for many choices of the game parameters and initial conditions, the equilibrium with lower value of the cooperation level is dynamically selected instead of the full cooperation one.

The results presented in this paper allow for a complete characterization of the shared reward dilemma in the following terms. Cooperation does not appear until the reward increases above the threshold $\delta = \min\{1, 1/n\zeta\}$. Interestingly, for $\delta > 1/n\zeta$, even a single cooperator can spread cooperation in the population, the more the larger the reward. This is an important point supporting the effectiveness of the reward mechanism for promoting the emergence of cooperation (Jiménez et al., 2007). Subsequently, for $\zeta \geqslant 1/2$ the fraction of cooperators increases monotonically until full cooperation is reached for $\delta = 1$. However, and quite unexpectedly, for $\zeta < 1/2$ an interesting phenomenon is observed: starting with a single cooperator, full invasion of the population only takes place when the scaled reward $\delta > \delta_c$, for some $\delta_c > 1$. This resistance to cooperation is remarkable because for $\delta > 1$ full cooperation is a stable equilibrium of the dynamics, and agrees with the dynamical analysis that shows that full cooperation is only reached if the initial fraction of cooperators is already large. When crossing δ_c cooperation suddenly invades. At that point, if we decrease the reward again, full cooperation persists down to $\delta = 1$. A slight decrease below this point produces an abrupt spread of defection in the population, which can even be completely invaded if $\zeta \leqslant 1/n$. This hysteresis is typical of critical phenomena, and it is very striking to find it in a model like this, where naïve intuition says that the more one rewards cooperation, the more cooperators should appear. The general, most important conclusion that can be drawn from this picture is that the effects of rewarding cooperation are neither trivial nor as straightforward as might be intuitively expected, and demand a more careful analysis. The origin of this complexity lies in the dilemma that the players confront and the impossibility to know *a priori* how much reward a player can get by cooperating.

One important issue for the shared reward dilemma is where this reward comes from. In Section 1 we have mentioned situations in biology that can fit the setup of the shared reward dilemma, as well as mechanisms of direct rewarding to foster more social behavior. To name just one, companies have realized the need of searching for mechanisms that motivate, provide incentives, or encourage cooperative behavior among their employees in order to contribute to the effective success of the teamwork. This context leads to another variant that we have not considered here: the case in which the reward is detracted from the payoff of all players. This case is particularly interesting for two reasons: first of all, for the feedback mechanism that it implies, and secondly, because it models a common scenario of taxation and subsequent subsidy of only certain people. Given the complexity of the shared reward game as we have analyzed it here, the results of this new scenario are presumed very rich. This tax-subsidy scenario has already been explored by some of us (Lugo and Jiménez, 2006) in a spatial evolutionary setup, but further, more detailed research is needed in view of the present findings. This issue will be the subject of a forthcoming work.

In closing, we have shown that rewarding introduces a new social dilemma. Depending on the parameters, the game casts the classical scenarios of full defection, coexistence of cooperators and defectors, bi-stability of full defection and full cooperation, or full cooperation, as well as more complex scenarios with two interior mixed equilibria, where bi-stability between a mixed equilibrium and full cooperation can occur. In addition, we have seen that the cooperative response may not be continuous on the reward, implying that promoting cooperation may require substantial incentives. We have shown that the classical (static) analysis of the game requires an evolutionary (dynamic) counterpart: while in the static case the counter-intuitive phenomenon of the decrease of the cooperation level upon increasing the reward may occur, this is never found dynamically; on the other hand, in the evolutionary framework we observe that very large rewards may be needed to establish a significant cooperation level, but once it is established, the reward may be very much reduced without damage to the cooperative behavior. Therefore, our general conclusion is that promoting cooperation through a reward mechanism is far from trivial, in agreement with the nontrivial behavior found in many social contexts, and deserves careful consideration prior to, and during, application.

Appendix A. Characterization of symmetric Nash equilibria

Theorem 1. *Let $\delta = \rho/n(n-1)(T-R)$ be the scaled reward of the game and $\zeta = (T-R)/(P-S)$ the defection ratio. Then, the following three scenarios can be found for the symmetric Nash equilibria of the shared reward dilemma with a number of players $n \geqslant 3$:*

1. *For $\zeta \geqslant 1/2$,*
   (i) *if $\delta \leqslant 1/n\zeta$, the unique Nash equilibrium is full defection ($q = 0$);*
   (ii) *if $1/n\zeta < \delta \leqslant 1$, the symmetric Nash equilibrium is a continuous function of δ which increases from 0^+ to 1, corresponding to the unique solution on $(0,1]$ of*

$$(\zeta - 1)x + 1 - \delta\zeta \frac{1-(1-x)^n}{x} = 0; \quad (A.1)$$

   (iii) *if $\delta > 1$ the unique Nash equilibrium is full cooperation ($q = 1$).*
2. *For $1/n \leqslant \zeta < 1/2$,*
   (i) *if $\delta \leqslant 1/n\zeta$ the only Nash equilibrium is full defection;*
   (ii) *if $1/n\zeta < \delta < 1$ the symmetric Nash equilibrium is a continuous function of δ which increases from 0^+ to some limit smaller than 1, corresponding to the unique solution on $(0,1)$ of (A.1);*
   (iii) *if $\delta \geqslant 1$ there exists $\delta_c > 1$ such that if $\delta > \delta_c$ the unique Nash equilibrium is $q = 1$, whereas if $1 \leqslant \delta \leqslant \delta_c$ there are two additional symmetric Nash equilibria corresponding to the solutions $0 < q_1 \leqslant q_2 \leqslant 1$ of (A.1) (equality, $q_1 = q_2$ holds only for $\delta = \delta_c$). The equilibria q_1 and q_2 are continuous monotone functions of δ (increasing and decreasing, respectively) and $q_2 = 1$ when $\delta = 1$.*
3. *For $\zeta < 1/n$,*
   (i) *if $\delta < 1$ the only Nash equilibrium is full defection;*
   (ii) *if $1 \leqslant \delta < 1/n\zeta$ the symmetric Nash equilibria are full defection, full cooperation and the unique solution on $(0,1]$ of (A.1), which is a continuous function of δ which decreases from 1 to some limit greater than 0;*
   (iii) *if $\delta \geqslant 1/n\zeta$ there exists $\delta_c > 1/n\zeta$ such that if $\delta > \delta_c$ the unique Nash equilibrium is $q = 1$, whereas if $1 \leqslant \delta \leqslant \delta_c$ there are two additional symmetric Nash equilibria corresponding to the solutions $0 \leqslant q_1 \leqslant q_2 < 1$ of (A.1) (equality, $q_1 = q_2$ holds only for $\delta = \delta_c$). The equilibria q_1 and q_2 are continuous monotone functions of δ (increasing and decreasing, respectively) and $q_1 = 0$ when $\delta = 1/n\zeta$.*

An upper bound for δ_c is given by

$$\delta_c \leqslant \frac{1}{4\zeta}\frac{n}{(n-1)}\frac{\left(1+\dfrac{2\zeta}{n-1}\right)^2}{\left(1-\dfrac{n-2}{n-1}\zeta\right)}. \quad (A.2)$$

Proof. As we discussed in Section 2, full cooperation is a Nash equilibria iff $\delta \geqslant 1$ and full defection is iff $\delta \leqslant 1/n\zeta$. To consider the remainder cases, let us define the "loss function" $\phi : [0,1] \to \mathbb{R}$,

$$\phi(x) = \frac{f_D(x) - f_C(x)}{(n-1)(P-S)} = \phi_1(x) - \delta\zeta\phi_2(x), \quad (A.3)$$

where $\phi_1(x) = x(\zeta - 1) + 1$ and

$$\phi_2(x) = n\mu_{n-1}(1,x) = \begin{cases} n & \text{for } x = 0, \\ \dfrac{1-(1-x)^n}{x} & \text{for } 0 < x \leqslant 1. \end{cases} \quad (A.4)$$

(c.f. Eqs. (4)–(6)). First of all, for $\delta = 0$ the only root of the loss function is at $x = 1/(1-\zeta)$, which, for any $\zeta > 0$, is outside the interval $[0,1]$. Hence $\phi(x) > 0$ for all $x \in [0,1]$ and the only Nash equilibrium if full defection. Let us henceforth assume $\delta > 0$. Function $\phi_2(x)$ decreases monotonically with x and, for any $n > 2$, is strictly convex within the interval $[0,1]$; instead, $\phi_1(x)$ is a straight line with nonnegative or negative slope depending on whether $\zeta \geqslant 1$ or $\zeta < 1$, respectively. For reasons that will be clear in a while, we need to consider separately the cases $\zeta \geqslant 1$, $\zeta < 1/n$ and $1/n \leqslant \zeta < 1$.

Case $\zeta \geqslant 1$: As $\phi_1(x)$ is nondecreasing, the loss function $\phi(x)$ monotonically increases with x and the only symmetric Nash equilibrium depends on the signs of $\phi(0) = 1 - \delta\zeta n$ and $\phi(1) = (1-\delta)\zeta$.

  (i) If $\delta \leqslant 1/n\zeta$ we have $0 \leqslant \phi(0) < \phi(1)$ and the unique Nash equilibrium is full defection. This equilibrium is strict for $\delta < 1/n\zeta$.
  (ii) If $1/n\zeta < \delta < 1$ we have $\phi(0) < 0$ and $\phi(1) > 0$, and the symmetric Nash equilibrium in mixed strategies is the solution $0 < q < 1$ of (A.1). Note that $\phi(x)$ decreases with δ, thus q increases with δ.
  (iii) If $\delta \geqslant 1$ we have $\phi(0) < \phi(1) \leqslant 0$ and the unique Nash equilibrium is full cooperation, which is strict for $\delta > 1$.

In the next two cases $\zeta < 1$ and therefore both $\phi_1(x)$ and $\phi_2(x)$ are decreasing functions of x. As $\phi_2(x)$ is convex, the situations that can occur are all sketched in Fig. A.1.

Case $\zeta < 1/n$: (i) If $\delta < 1$ then $\phi(0) > 0$ and $\phi(1) > 0$ and we have the situation sketched in Fig. A.1(a). The only Nash equilibrium is full defection.

(ii) If $1 \leqslant \delta < 1/n\zeta$ we have $\phi(0) > 0$ and $\phi(1) \leqslant 0$, so the situation is as sketched in Fig. A.1(b) and therefore there will be a symmetric equilibrium $0 < q \leqslant 1$. Note that $q = 1$ for $\delta = 1$ and decreases as δ goes to $1/n\zeta$.
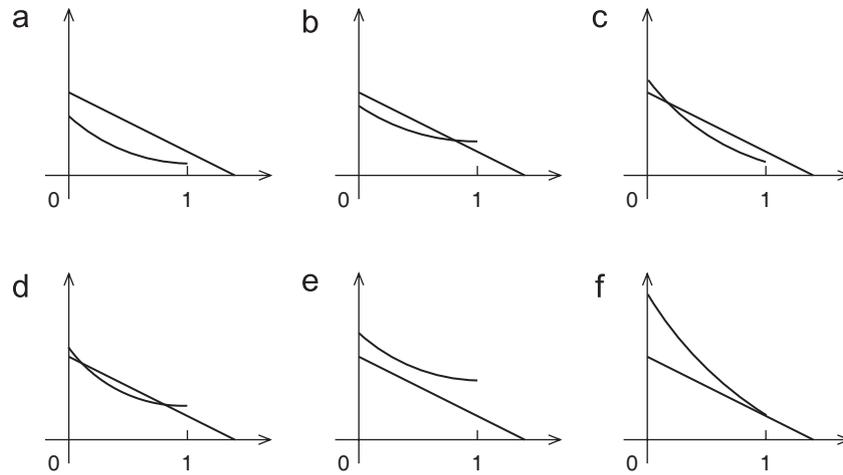
Fig. A.1. Relative situations of $\phi_1(x)$ and $\delta\zeta\phi_2(x)$ (see text).

(iii) If $1/n\zeta \leqslant \delta$ then $\phi(0) \leqslant 0$ and $\phi(1) < 0$. Thus we will have one of the two situations plotted in Figs. A.1(d) and (e) depending on the slopes of $\phi_1(x)$ and $\phi_2(x)$ at $x = 0$ at the crossover $\delta = 1/n\zeta$, where $\phi(0)$ changes sign. If $\phi_1'(0) > \phi_2'(0)/n$ the situation will be as illustrated in Fig. A.1(d), and if $\phi_1'(0) \leqslant \phi_2'(0)/n$ it will be as in Fig. A.1(e). In the former case there will be two Nash equilibria, $0 < q_1 < q_2 < 1$, and in the latter the only Nash equilibrium will be $q = 1$. As $\phi_1'(x) = \zeta - 1$ and

$$\phi_2'(x) = \frac{nx(1-x)^{n-1} - 1 + (1-x)^n}{x^2}, \tag{A.5}$$

we have $\phi_1'(0) = \zeta - 1$ and $\phi_2'(0) = -n(n-1)/2$. The condition $\phi_1'(0) > \phi_2'(0)/n$ reads $\zeta > (3 - n)/2$, which holds for any $n \geqslant 3$. We thus find two equilibria, $0 \leqslant q_1 < q_2 < 1$, which, upon increasing δ, approach each other (q_1 increases and q_2 decreases) up to δ_c, where they coalesce in one Nash equilibrium $q \in (0, 1)$. Finally, for $\delta > \delta_c$ the only Nash equilibrium is full cooperation.

Case $1/n \leqslant \zeta < 1$: (i) If $\delta < 1/n\zeta$ then $\phi(0) > 0$ and $\phi(1) > 0$ and we have the situation sketched in Fig. A.1(a). The only Nash equilibrium is again $q = 0$.

(ii) If $1/n\zeta \leqslant \delta < 1$ (this case is empty if $\zeta = 1/n$) then $\phi(0) \leqslant 0$ and $\phi(1) > 0$, and we have the situation depicted in Fig. A.1(c). There is a unique symmetric Nash equilibrium $q \in [0, 1)$ determined by (A.1). Also $q = 0$ for $\delta = 1/n\zeta$ and increases as δ goes to 1.

(iii) If $\delta \geqslant 1$ then $\phi(0) \leqslant 0$ and $\phi(1) \leqslant 0$. In this case we may have two additional equilibria if the situation of Fig. A.1(d) occurs, or just one if either $\delta > 1$ and we have the situation of Fig. A.1(e), or $\delta = 1$ and the situation is like in Fig. A.1(f). The separation between the first case and the last two cases depends on which scenario, Fig. A.1(d) or (f), we have at $\delta = 1$. This, in turn, depends on the slopes of $\phi_1(x)$ and $\phi_2(x)$ at $x = 1$ when $\delta = 1$: if $\phi_1'(1) < \zeta\phi_2'(1)$ then we will have Fig. A.1(d), and if $\phi_1'(1) \geqslant \zeta\phi_2'(1)$ we will have Fig. A.1(f). The former is equivalent to $\zeta < 1/2$, the latter to $\zeta \geqslant 1/2$. So if $\zeta \geqslant 1/2$ the only Nash equilibrium is

$q = 1$, whereas if $\zeta < 1/2$ there will be, for $1 \leqslant \delta < \delta_c$, two equilibria, $0 < q_1 < q_2 \leqslant 1$, which coalesce in a single one at $\delta = \delta_c$. For $\delta > \delta_c$ the only Nash equilibrium is $q = 1$.

The limiting value δ_c can be determined as the value of δ at which the curve $\phi_1(x)$ is tangent to $\delta_c\zeta\phi_2(x)$ at a point $x_c \in (0, 1)$. At this point the two equations

$$\phi_1(x_c) = \delta_c\zeta\phi_2(x_c), \quad \phi_1'(x_c) = \delta_c\zeta\phi_2'(x_c), \tag{A.6}$$

hold simultaneously. These two equations can be combined to yield

$$\delta_c\zeta(1-x)^n = x_c^2(1-\zeta) - x_c + \delta_c\zeta, \tag{A.7}$$

$$[(n-1) - (n-2)\zeta]x_c^2 - (n-1+2\zeta)x_c + \delta_c\zeta n = 0. \tag{A.8}$$

For x_c to exist it is necessary that the second equation has a solution. The condition for this to happen is

$$(n-1+2\zeta)^2 - 4[(n-1) - (n-2)\zeta]\delta_c\zeta n \geqslant 0. \tag{A.9}$$

Since $\zeta < 1/2$ then $(n-1) - (n-2)\zeta > 0$, so the above equation holds provided

$$\delta_c \leqslant \frac{(n-1+2\zeta)^2}{4[(n-1) - (n-2)\zeta]\zeta n} = \frac{\left(1 + \dfrac{2\zeta}{n-1}\right)^2}{4\zeta\left(1 - \dfrac{n-2}{n-1}\zeta\right)}\left(\frac{n-1}{n}\right). \tag{A.10}$$

This expresses an upper bound for δ_c.   □

Corollary 1. *Consider a sequence* $\{\rho_n\}$ *of rewards such that* $\rho_n \to \infty$ *as* $n \to \infty$ *in such a way that*

$$\delta = \lim_{n \to \infty} \frac{\rho_n}{n^2(T - R)}, \tag{A.11}$$

with $0 \leqslant \delta < \infty$. *Let us define* $\delta_\zeta = 1/4\zeta(1 - \zeta)$. *Then, in the limit* $n \to \infty$, *the Nash equilibria of the shared reward dilemma are*

(i) *full defection if* $\delta = 0$;

(ii) *a unique equilibrium in mixed strategies*

$$q = \frac{1 - \sqrt{1 - \delta/\delta_\zeta}}{2(1 - \zeta)} \qquad (A.12)$$

if $0 < \delta < 1$;

(iii) *full cooperation and two equilibria in mixed strategies,* $0 < q_1 \leqslant q_2 < 1$, *where* q_1 *is given by* (A.12) *and*

$$q_2 = \frac{1 + \sqrt{1 - \delta/\delta_\zeta}}{2(1 - \zeta)}, \qquad (A.13)$$

if $1 < \delta \leqslant \delta_\zeta$ *and* $\zeta < 1/2$ *(equality* $q_1 = q_2 = 1/2(1 - \zeta)$ *only holds if* $\delta = \delta_\zeta$*), and*

(iv) *full cooperation otherwise.*

Proof. 1. As $n \to \infty$ only two of the three cases of Theorem 1 remain, corresponding now to $\zeta \geqslant 1/2$ and $0 \leqslant \zeta < 1/2$. Besides, Eq. (A.1) becomes the quadratic equation

$$(\zeta - 1)x^2 + x - \delta\zeta = 0, \qquad (A.14)$$

whose two solutions are

$$q_1 = \frac{1 - \sqrt{1 - 4\zeta(1 - \zeta)\delta}}{2(1 - \zeta)}, \quad q_2 = \frac{1 + \sqrt{1 - 4\zeta(1 - \zeta)\delta}}{2(1 - \zeta)}. \qquad (A.15)$$

Both are real whenever $0 \leqslant \delta \leqslant \delta_\zeta = 1/4\zeta(1 - \zeta)$. On the other hand, q_1 monotonically increases with δ. If $\zeta \geqslant 1/2$, q_1 runs from 0 to 1 as δ moves from 0 to 1; if $\zeta < 1/2$, q_1 goes from 0 to $1/2(1 - \zeta)$ as δ goes from 0 to δ_ζ. As for q_2, the condition for it to be within the interval $[0, 1]$ is $\zeta \leqslant 1/2$ and $1 \leqslant \delta \leqslant \delta_\zeta$. When $\zeta = 1/2$ and $\delta = 1$ then $q_2 = q_1 = 1$. When $\zeta < 1/2$ then q_2 provides a second solution, monotonically decreasing from 1 down to $1/2(1 - \zeta)$ as δ runs from 1 to δ_ζ, where it coalesces with q_1.

Finally, for $\delta > \delta_\zeta$ we have

$$(\zeta - 1)x^2 + x - \delta\zeta > 0, \qquad (A.16)$$

so the only Nash equilibrium is full cooperation. $\square$

Appendix B. Characterization of asymmetric Nash equilibria

Theorem 2. *Let* $\delta = \rho/n(n - 1)(T - R)$ *be the scaled reward of the game and* $\zeta = (T - R)/(P - S)$ *the defection ratio. Let*

$$\delta_k = k\frac{n - 1 + (k - 1)(\zeta - 1)}{n(n - 1)\zeta}, \quad k = 1, 2, \ldots, n - 1. \quad (B.1)$$

Then a configuration with $1 \leqslant k \leqslant n - 1$ *cooperators and* $n - k$ *defectors will be a Nash equilibrium in pure strategies of the shared reward dilemma if and only if* $\delta_k \leqslant \delta \leqslant \delta_{k+1}$ *and, when* $\zeta < 1/2$, $k \leqslant (n - 1)/2(1 - \zeta)$.

Proof. According to (1), in a configuration with k cooperators and $n - k$ defectors the payoff of a cooperator is

$$\mathscr{P}_C(k) = (k - 1)R + (n - k)S + \frac{\rho}{k} \qquad (B.2)$$

and of a defector

$$\mathscr{P}_D(k) = kT + (n - 1 - k)P. \qquad (B.3)$$

For such a configuration to be a Nash equilibrium in pure strategies two requirements must be met: (i) a cooperator cannot get higher payoff by defecting, and (ii) a defector cannot get a higher payoff by cooperating. Condition (i) amounts to saying that $\mathscr{P}_C(k) - \mathscr{P}_D(k - 1) \geqslant 0$, i.e.

$$(k - 1)(T - R) + (n - k)(P - S) - \frac{\rho}{k} \leqslant 0, \qquad (B.4)$$

and condition (ii) amounts to saying that $\mathscr{P}_D(k) - \mathscr{P}_C(k + 1) \geqslant 0$, i.e.

$$k(T - R) + (n - 1 - k)(P - S) - \frac{\rho}{k + 1} \leqslant 0. \qquad (B.5)$$

By defining the parabola

$$\psi(x) = x^2(\zeta - 1) + (n - \zeta)x - \Delta, \qquad (B.6)$$

where $\Delta = \rho/(P - S) = n(n - 1)\delta\zeta$, and taking into account that $P - S > 0$, the two conditions above can be rewritten

$$\psi(k) \leqslant 0, \quad \psi(k + 1) \geqslant 0. \qquad (B.7)$$

In other words, an asymmetric Nash equilibrium in pure strategies exists if and only if there exists $k = 1, 2, \ldots, n - 1$ such that (B.7) holds.

The two roots of the parabola (B.6) are

$$x_\pm = \frac{-(n - \zeta) \pm \sqrt{(n - \zeta)^2 + 4\Delta(\zeta - 1)}}{2(\zeta - 1)}, \qquad (B.8)$$

so for the discussion to follow we should treat separately the cases $\zeta > 1$, $\zeta = 1$, and $\zeta < 1$.

Case $\zeta > 1$: In this case the parabola is convex, both roots are real and $x_- < 0$ and $x_+ > 0$. So there will be an asymmetric Nash equilibrium in pure strategies with k cooperators if and only if $k \leqslant x_+ \leqslant k + 1$, i.e.

$$2k(\zeta - 1) \leqslant \sqrt{(n - \zeta)^2 + 4\Delta(\zeta - 1)} - (n - \zeta)$$
$$\leqslant 2(k + 1)(\zeta - 1) \qquad (B.9)$$

or equivalently

$$(2k - 1)\zeta + n - 2k \leqslant \sqrt{(n - \zeta)^2 + 4\Delta(\zeta - 1)}$$
$$\leqslant (2k + 1)\zeta + n - 2(k + 1). \qquad (B.10)$$

As $\zeta > 1$ we have $(2k - 1)\zeta + n - 2k > n - 1 > 0$, so all three terms in (B.10) are positive numbers and can be squared to obtain, after simplifying,

$$k[n - k + (k - 1)\zeta] \leqslant \Delta \leqslant (k + 1)(n - k - 1 + k\zeta). \qquad (B.11)$$

Given that $\Delta = n(n - 1)\delta\zeta$, these inequalities can be rewritten as

$$\delta_k \leqslant \delta \leqslant \delta_{k+1}, \quad \delta_k \equiv k\frac{n - 1 + (k - 1)(\zeta - 1)}{n(n - 1)\zeta}. \qquad (B.12)$$

Notice that if $\zeta > 1$ then $\{\delta_k\}$ forms an increasing sequence and that $\delta_1 = 1/n\zeta$ and $\delta_n = 1$.

Case $\zeta = 1$: In this case only the root $x_0 = \Delta/(n-1) = n\delta$ exists, thus the condition $k \leqslant x_0 \leqslant k+1$ is equivalent to (B.12), where, of course, $\delta_k = k/n$.

Case $\zeta < 1$: The parabola (B.6) is now concave and the roots can be rewritten

$$x_{\pm} = \frac{(n-\zeta) \mp \sqrt{(n-\zeta)^2 - 4\Delta(1-\zeta)}}{2(1-\zeta)}. \tag{B.13}$$

For them to be real we must have

$$(n-\zeta)^2 - 4\Delta(1-\zeta) \geqslant 0. \tag{B.14}$$

Suppose this inequality holds; then we have $x_{\pm} > 0$ and $x_+ < x_-$. For an asymmetric Nash equilibrium with k cooperators to exist we must have $k \leqslant x_+ \leqslant k+1 \leqslant x_-$.

The inequalities $x_+ \leqslant k+1 \leqslant x_-$ are equivalent to

$$|n - 2k + (2k-1)\zeta| \leqslant \sqrt{(n-\zeta)^2 - 4\Delta(1-\zeta)}. \tag{B.15}$$

Squaring again this expression boils down to $\delta \leqslant \delta_{k+1}$. The inequality $k \leqslant x_+$ can be rewritten

$$\sqrt{(n-\zeta)^2 - 4\Delta(1-\zeta)} \leqslant n - 2k + (2k-1)\zeta. \tag{B.16}$$

No value of Δ satisfies this inequality unless the right-hand-side is nonnegative; in other words, unless

$$k \leqslant \frac{n-\zeta}{2(1-\zeta)}. \tag{B.17}$$

Assuming (B.17) holds we can square and simplify once more to get $\delta \geqslant \delta_k$.

But there is one last remark to make: $\delta_k \leqslant \delta \leqslant \delta_{k+1}$ is empty unless $\delta_k \leqslant \delta_{k+1}$. If $\zeta \geqslant 1$ then δ_k is an increasing sequence, but for $\zeta < 1$ this is no longer true, and the constraint $\delta_k \leqslant \delta_{k+1}$ implies

$$k \leqslant \frac{n-1}{2(1-\zeta)}, \tag{B.18}$$

which is more restrictive than (B.17). Notice that this only constraints the value of k provided $\zeta < 1/2$.

Finally, one can check that (B.14) holds for any δ_k because

$$(n-\zeta)^2 - 4n(n-1)\zeta(1-\zeta)\delta_k$$
$$= [(2k-1)(1-\zeta) - n + 1]^2 \geqslant 0. \quad \square \tag{B.19}$$

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