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Kink decay in a parametrically driven ϕ^4 chain

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We consider the parametrically driven discrete ϕ^4 model with loss. Using an analytical approach and direct numerical simulations, which turn out to be in excellent agreement, we demonstrate that there is a range of driving parameters such that kinks cannot exist in the system. This effect may be understood with the help of an averaged external potential energy of the system, which has no double-well structure in this region.

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I. INTRODUCTION

The ϕ^4 model has become an important subject because of its numerous applications in condensed-matter physics. It describes, for example, structural phase transitions in ferroelectric and ferromagnetic materials [1]–[5], topological excitations in quasi-one-dimensional systems like biological macromolecules and hydrogen-bonded chains [6, 7], or polymers [8]–[10], etc. Its simplest localized solutions are the so-called kinks, which are related to the motion of the aforementioned topological excitations, e.g., domain walls in second-order phase transitions [1, 2] or polymerization mismatches [8]. A more realistic modeling of physical situations in condensed-matter physics often requires the inclusion of perturbations of different types, like thermal noise [4] and time [11–13] or spatial [14] -dependent potential fluctuations. These perturbations lead to a modification of the system parameters, and, in particular, most of them change the shape of the double-well potential of the model (see, e.g., Ref. [13] for a detailed description).

Considering fluctuations as a superposition of different harmonics with random amplitudes, it is natural to analyze the structural stability of kinks in such a periodically varying, double-well potential. The purpose of this paper is to demonstrate that there is a range of values of the parameters of a parametrically driven ϕ^4 model in which kinks cannot exist. This assertion means that the *averaged dynamics* of the wave field is described by a single-well potential instead of a double-well one. We predict kink anihilation analytically by means of an approach that is similar to the well-known method of averaging for a pendulum motion under a high-frequency parametric force [15] (see also Ref. [16]). Using numerical simulations of the so perturbed ϕ^4 model, we show that the kink decay due to the parametrical driving may be actually observed, and that there is an extremely good agreement between our analytical predictions and numerical results.

The paper is organized as follows. Section II contains analytical results obtained by averaging the fast oscillations in the perturbed ϕ^4 model. Section III presents results of direct numerical simulations. Section IV concludes the paper.

II. ANALYTICAL APPROACH

We consider a discrete nonlinear chain under the influence of a parametric force and damping, whose corresponding evolution equation is

$$\frac{d^2\phi_n}{dt^2} - K\,(\phi_{n+1} - 2\phi_n + \phi_{n-1}) + \frac{dV(\phi_n)}{d\phi_n} = a\,G(\phi_n)\,\cos\omega t - \gamma\frac{d\phi_n}{dt}, \quad (1)$$

where a and ω are the amplitude and frequency of the periodic force and γ is a damping coefficient. On the other hand, the parameter K accounts for the coupling between particles, while $V(\phi_n)$ is the external (substrate) potential and $G(\phi_n)$ characterizes the coupling between the wave field and the force. For the sake of definiteness, we will subsequently take

$$V(\phi) \equiv \frac{1}{4}\,(1-\phi^2)^2, \quad (2)$$

and with respect to the coupling, we will let

$$G(\phi) \equiv \phi. \quad (3)$$

In the absence of perturbation, Eq. (1) with the choice in Eq. (2) is nothing but a chain of particles with nearest-neighbor interactions, each one of them on an on-site, double-well potential. Substituting the ansatz $\phi_n = \pm 1 + \phi_0\,\exp(iqnb - i\Omega t)$ for the solution of Eq. (1), having in mind (2), letting $a = \gamma = 0$, and expanding in small ϕ_0, we find that small-amplitude excitations around the two minima $\phi_n = \pm 1$ are described by waves with wave number q and frequency Ω, obeying a dispersion law of the form

$$\Omega^2 = 2 + 4K\sin^2\left(\frac{qb}{2}\right), \quad (4)$$

which amounts to saying that eigenfrequencies lie in a bounded region, $\Omega^2_{\rm min} \le \Omega^2 \le \Omega^2_{\rm max}$, where $\Omega^2_{\rm min} \equiv 2$, $\Omega^2_{\rm max} \equiv 2 + 4K$.

In the continuum limit, when $qb \ll 1$, Eq. (4) yields the dispersion law $\Omega^2 = 2 + Kq^2b^2$, and Eqs. (1) and (2) with $a = \gamma = 0$ are transformed into the well known ϕ^4 model,

$$\phi_{tt} - Kb^2\phi_{xx} - \phi + \phi^3 = 0, \tag{5}$$

where the variable $x \equiv nb$ is considered to be continuous. As is well known (see, e.g., Ref. [2]), Eq. (5) has an exact kink solution that connects two parts of the chain at different minima of the potential (2), i.e., at $\phi_n = \pm 1$. The static form of the kink is given by

$$\phi = \pm \tanh\left(\frac{x}{b\sqrt{2K}}\right), \tag{6}$$

where the signs + or − stand for the kink and antikink, respectively. In the discrete chain, the kink (6) may be considered as an approximate solution, which is stable due to topological properties; in such a case, the discrete value nb describes the positions of the particles.

We will consider now the general expression, Eq. (1), without specifying any particular choice for V and G, so as to lose no generality at all. We will study this perturbed system assuming that the driving frequency ω is larger than the limit frequency $\Omega_{\max}$. To describe the corresponding nonlinear dynamics under the influence of such a parametric force, we will apply an averaging method analogous to the well-known one for the Kapitza problem, i.e., for the dynamics of a pendulum hanging from an oscillating suspension point (see Ref. [15]). In order to derive an averaged equation of motion in our case, we will decompose the wave field ϕ_n into a sum of slow and fast varying parts, that is to say,

$$\phi_n(t) = \Phi_n(t) + \xi_n(t), \tag{7}$$

where the functions $\Phi_n(t)$ and $\xi_n(t)$ describe the slow and fast evolutions, respectively. The function $\xi_n(t)$ stands for small oscillations around the slowly varying field $\Phi_n(t)$, and the mean value of $\xi_n(t)$ during the period $2\pi/\omega$ is assumed to be zero, i.e., $\langle\phi_n(t)\rangle = \Phi_n(t)$, the brackets standing for time average. Our goal is to derive an effective equation for the function $\Phi_n(t)$ that describes the slow ("averaged") wave field. To this end, we substitute Eq. (7) into Eq. (1), and expanding in ξ_n, which we assume to be small enough for such a purpose, we obtain the equation

$$\ddot{\Phi}_n + \ddot{\xi}_n - K(\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}) - K(\xi_{n+1} - 2\xi_n + \xi_{n-1}) + V'(\Phi_n) + \xi_n V''(\Phi_n) + \frac{1}{2}\xi_n^2 V'''(\Phi_n)$$
$$= a\, G(\Phi_n)\, \cos\omega t + a\, G'(\Phi_n)\, \xi_n\, \cos\omega t - \gamma\dot{\Phi}_n - \gamma\dot{\xi}_n, \tag{8}$$

where we have neglected higher-order term contributions.

The periodic driving $a\, G(\phi)\, \cos\omega t$ looks like a parametric force term. However, in the asymptotic parts of the kink (or kink tails), i.e., n values such that $\phi = \pm 1$, it simply acts as an external periodic force. According to Eq. (4), this assertion means that if the frequency lies outside the eigenfrequency band $(\Omega_{\min}, \Omega_{\max})$ (this must be compared with the requirement $\Omega_{\min} < \Omega/2 < \Omega_{\max}$ for a parametric force), the periodic forcing cannot excite linear waves in the system. Then, it is natural to assume that the function $\xi_n(t)$ is a slow function of n and write $\xi_n(t) \approx \xi_{n\pm 1}(t)$.

Let us go on taking a careful look at Eq. (8). It is clear that it has terms of a very different nature, slow and fast varying ones. Hence, these different terms must verify the equality in Eq. (8) *separately*, giving rise to their own particular equations. In order to satisfy the rapidly oscillating part of Eq. (8), it is necessary to take into account all terms which are proportional to the rapidly varying function $\xi_n(t)$ plus the term $\sim a\cos\omega t$, which is also fast. As a result, the following equation must hold (recall again that we assume $\xi_{n\pm 1} \approx \xi_n$; equivalently, there is no excitation of linear waves by the oscillating force)

$$\ddot{\xi}_n + \xi_n V''(\Phi_n) = a\, G(\Phi_n)\, \cos\omega t - \gamma\dot{\xi}_n. \tag{9}$$

In principle, the second term in the left-hand side (lhs) of Eq. (9) is smaller than the first one, because $\ddot{\xi}_n$ is proportional to the large value ω^2. However, assuming that the theory may apply to the case when $\omega^2 > \Omega_{\max}^2$ but when ω is not very large, we will keep this term. Besides, we may also think of including in Eq. (9) higher-order terms, i.e., $\sim \xi_n^2$ and $\sim a\xi_n\cos\omega t$, but these terms have an additional small multiplier ξ_n in comparison with those appearing in Eq. (9). Moreover, they are more important in the equation for the slowly varying field $\Phi_n(t)$, where they contribute because $\langle\xi_n^2\rangle$ and $\langle\xi_n\cos\omega t\rangle$ are nonzero. Finally, as may be also seen from Eq. (9), there is no limitation for the dissipative coefficient γ, which may also be not so small.

Now, as Φ_n evolves much slower than ξ_n, we can consider all functions of Φ_n as constants in time, and subsequently write the forced solution of Eq. (9) in the form [15]

$$\xi_n(t) = \frac{a\, G(\Phi_n)}{\sqrt{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}}\, \cos(\omega t + \delta), \tag{10}$$

where

$$\cos\delta = -\frac{(\omega^2 - \omega_0^2)}{\sqrt{(\omega^2 - \omega_0^2) + \gamma^2\omega^2}}, \tag{11}$$

and

$$\omega_0^2 \equiv V''(\Phi_n). \tag{12}$$

The next stage is to input Eqs. (10) and (11) into Eq. (8) and to average over the fast oscillations. By so doing we derive the equation for the slowly varying function $\Phi_n(t)$, which turns out to be

$$\ddot{\Phi}_n+\gamma\dot{\Phi}_n-K(\Phi_{n+1}-2\Phi_n+\Phi_{n-1})+V'(\Phi_n)=-\frac{a^2G^2(\Phi_n)V'''(\Phi_n)}{4[(\omega^2-\omega_0^2)^2+\gamma^2\omega^2]}-\frac{a^2(\omega^2-\omega_0^2)G'(\Phi_n)G(\Phi_n)}{2[(\omega^2-\omega_0^2)^2+\gamma^2\omega^2]}, \quad (13)$$

with ω_0^2 defined as in Eq. (12); on the other hand, we have used the results

$$\langle\xi_n^2(t)\rangle=\frac{a^2G^2(\Phi_n)}{2[(\omega^2-\omega_0^2)^2+\gamma^2\omega^2]}, \quad (14)$$

$$\langle a\,\xi_n(t)\,\cos\omega t\rangle=\frac{a^2G(\Phi_n)\,\cos\delta}{2\sqrt{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}}=-\frac{a^2G(\Phi_n)(\omega^2-\omega_0^2)}{2[(\omega^2-\omega_0^2)^2+\gamma^2\omega^2]}. \quad (15)$$

In the case when $\omega^2\gg\omega_0^2$, i.e., in fact, in the region $\omega^2\gg\Omega_{\rm max}^2$, the first term in the right-hand side (rhs) of Eq. (13) can be omitted, yielding

$$\ddot{\Phi}_n-K(\Phi_{n+1}-2\Phi_n+\Phi_{n-1})+\frac{dV_{\rm eff}}{d\Phi_n}=-\gamma\dot{\Phi}_n, \quad (16)$$

where we have defined

$$V_{\rm eff}(\Phi)\equiv V(\Phi)+\frac{a^2G^2(\Phi)}{4(\omega^2+\gamma^2)}. \quad (17)$$

Let us now particularize our results for the ϕ^4 model, recalling Eq. (2) with the parametric force given by Eq. (3) to get the corresponding effective potential,

$$V_{\rm eff}(\Phi)=\frac{1}{4}\left(1-2\Phi_{\rm min}^2\Phi^2+\Phi^4\right), \quad (18)$$

$$\Phi_{\rm min}\equiv\left(1-\frac{a^2}{2(\omega^2+\gamma^2)}\right)^{1/2}. \quad (19)$$

Consequently, the potential (18) will have a double-well shape, provided that

$$a^2<2(\omega^2+\gamma^2); \quad (20)$$

under this condition, the perturbed system can support kink excitations of the averaged field, whose expression, in the continuum limit and in their own rest reference frame, is [cf. Eq. (5)]

$$\Phi(x,t)=\pm\Phi_{\rm min}\,\tanh\left(\frac{\Phi_{\rm min}x}{b\sqrt{2K}}\right). \quad (21)$$

According to Eqs. (19) and (21), the kink amplitude $2\Phi_{\rm min}$ is a function of the force parameters, so that it may be changed by the force. If $\Phi_{\rm min}\to 0$, the kink width goes to infinity and the kink itself disappears. Comparing Eq. (21) and Eq. (6), one might initially expect that the modified kink (21) corresponds to an infinite energy situation because the asymptotes are not ±1 as before. This can be understood by recalling that the periodic force must be applied to *all of the particles in the chain*, and, therefore, the parametrically driven model makes sense only for finite system lengths L; having this in mind, the approach we use in this paper is valid, provided the kink width ($\sim b\sqrt{K}/\Phi_{\rm min}$) is much less than L.

If the condition (20) no longer holds, and $a^2\geq 2(\omega^2+\gamma^2)$, the effective potential (18) does not have a double-well structure and kinks cannot exist. The condition (20) may be transformed in another one for the frequency ω of the parametric force, assuming also that ω is larger than the maximum eigenfrequency of the system $\Omega_{\rm max}$,

$$\omega^2>\frac{a^2}{2}-\gamma^2. \quad (22)$$

In the general case, when $\omega^2\sim\omega_0^2$, the first and second terms in the rhs of Eq. (13) are of the same order, and we cannot neglect any of them; subsequently, we cannot compute the effective potential either. Nevertheless, in the particular problem of the ϕ^4 model, $V(\Phi)$ and $G(\Phi)$ are given by simple expressions and we can write the effective force explicitly as follows:

$$F_{\rm eff}(\Phi)\equiv V'_{\rm eff}(\Phi)=-\Phi+\Phi^3+\frac{a^2(\omega^2+1)\,\Phi}{2[(\omega^2-3\Phi^2+1)^2+\gamma^2\omega^2]}. \quad (23)$$

Equating Eq. (23) to zero, we may find the extrema of the effective potential $V_{\rm eff}(\Phi)$. Thus $\Phi=0$ is always an extremum, and other extrema, $\pm\Phi_{\rm min}$, are given by the equation

$$\Phi_{\rm min}^2=1-\frac{a^2(\omega^2+1)}{2[(\omega^2-3\Phi_{\rm min}^2+1)^2+\gamma^2\omega^2]}. \quad (24)$$

These two additional extrema exist when $\Phi_{\rm min}^2>0$. Hence, the critical condition corresponds to $\Phi_{\rm min}^2=0$, which means that the extrema exist, provided

$$a^2<\frac{2[(\omega^2+1)^2+\gamma^2\omega^2]}{(\omega^2+1)}. \quad (25)$$

It can be easily shown that if $\omega^2\gg1$ the previous condition (20) is recovered from Eq. (25).

To conclude this section, we would like to make some comments related to the possibility of chaotic regimes happening in the chain. Equation (9) describes, in fact, the chain behavior far from the kink; however, as is well known, such a homogeneous oscillatory dynamics may show chaotic evolution when the amplitude is large enough, which implies that nonlinear terms must be taken into account. This means that the above-presented analytical approach is valid only outside this chaotic region. To estimate the range of the system parameters where chaos can be observed, we will use some considerations based on the Melnikov function [17]. According to Eq. (1), the homogeneous oscillations of the ϕ^4 chain under the parametric perturbation are described by the equation

$$\ddot{\xi}-\xi+\xi^3=a\,\xi\,\cos(\omega t)-\gamma\dot{\xi}. \quad (26)$$

Equation (26) is similar to the one considered in Ref. [18], where the parametric modulation affected only the cubic term. Calculation of the Melnikov function $\Delta(t_0)$ for our problem, Eq. (26), yields (see details of similar computations in, e.g., Ref. [18] and references therein)

$$\Delta(t_0) = \frac{\pi a(\omega^2 - 1)}{\sinh(\pi\omega/2)}\sin(\omega t_0) + \frac{4\gamma}{3}. \tag{27}$$

If the parameters are such that the function $\Delta(t_0)$ defined in Eq. (27) changes sign, then homoclinic intersections are present, and hence chaotic solutions of Eq. (26) exist. Nevertheless, $\Delta(t_0)$ will not change sign (which prevents the onset of chaos), provided the inequality

$$\gamma > \frac{3\pi a\,(\omega^2 - 1)}{4\sinh(\pi\omega/2)} \tag{28}$$

holds. As can be seen from this equation, large frequencies (like the ones we are concerned with) easily fulfill this requirement, provided there is some dissipation acting in the system, as indeed it does.

III. NUMERICAL RESULTS

In order to get a deeper insight into the system (1) and check our analytical approach validity, we have performed a number of direct numerical simulations of the system for the ϕ^4 case, i.e., Eq. (2) and driving as in Eq. (3). The basis of our numerical procedure is the Strauss-Vázquez finite-difference scheme [19]. The advantages of this method are that it has been proved to be stable and convergent [20] and to be free of numerical blow ups [21] in the unperturbed case. Moreover, it has been used to study many perturbed nonlinear Klein-Gordon problems (see, e.g., Refs. [12]-[14]), with very good results.

The scheme has been thoroughly described elsewhere (cf. references in the previous paragraph) and we are not going to repeat it here; however, it is worth explaining some points on this specific case. First of all, the size of the lattice parameter Δx must be chosen to be neither too large as to induce discreteness effects that disturb kink propagation nor too small as to transform the model into a quasicontinuum one, with the subsequent widening of the eigenfrequency spectrum $[\Omega_{\min}, \Omega_{\max}]$ (notice that $\Omega_{\max}$ goes to infinity when Δx goes to zero). We have accomplished this requirement by letting $\Delta x = 0.1$, which in turn implies $\Omega^2_{\max} = 402$ ($\Omega_{\max} \simeq 20$). Secondly, the choice for the lattice spacing poses a constraint on our time step to integrate Eq. (1) because the stability of the Strauss-Vázquez scheme is guaranteed only if $\Delta t \leq \Delta x$. Furthermore, to reproduce faithfully the parametric driving for high frequencies ω it is necessary to have $\Delta t \ll \omega^{-1}$. We have then chosen $\Delta t = 0.01$ for almost all the simulations (for frequencies near $\omega = 50$ it was necessary to take $\Delta t = 0.005$ or even $\Delta t = 0.0025$) and we have checked that the results did not change upon the decreasing of this time step; in this way we have been able to ensure that the outcome of our numerical computations makes physical sense.

Our simulations have been carried out on a system of $N = 401$ particles with the initial condition given by a ϕ^4 kink centered at $n = 201$, Eq. (4), with tails of value $\phi = \pm 1$. To allow its evolution and subsequent decay, if any, we have imposed antiperiodic boundary conditions at the edges of the system, i.e., $\phi_{N+1} = -\phi_1$, $\phi_0 = -\phi_N$. Finally, we have introduced a somewhat large dissipation value, $\gamma = 10$, to avoid great amplitudes of the wave field and fast oscillations that can lead to chaotic regimes, as we have explained above [see Eq. (28)].

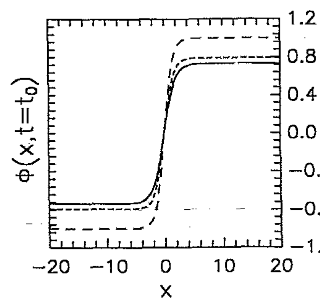

FIG. 1. Kink structure in the parametrically driven ϕ^4 chain when $a = 25$, $\omega = 25$, and $\gamma = 10$. The initial state is an unperturbed kink at rest (large-dashed line). Other shown profiles correspond to instants $t = 10$ (small-dashed line) and $t = 100$ (solid line); this last one is in the asymptotic state.

The results of our simulations are plotted in Figs. 1–6. As a first example, Figs. 1 and 2 show the effect of a parametric driving with $a = 25$, $\omega = 25$ on the initial kink. Recalling (20) and substituting for ω and γ we find that the approximate threshold above which the kink does not exist anymore is $a > \sqrt{1500} \simeq 38$. Accordingly, from the plots it appears distinctly that the kink still exists, though its amplitude has diminished from $\Phi_{\min} = 1$ at $t = 0$ [see Eq. (21)] to $\Phi_{\min} \sim 0.315$. This evolution is fast at early stages and slow as time becomes large, reaching an asymptotic state in which the kink oscillates around the bottom of the wells of the renormalized potential [see Fig. 2; we represent $\Phi_{\min}$ at the tails by the value of $\Phi(n = 301)$ or, in other words, $\Phi(x = 10)$]. If we increase the driving-force amplitude above the threshold, letting $a = 38$, Figs. 3 and 4, we appreciate that the kink actually decays for these parameters, again rapidly at first and asymptotically for large times.

The whole of our results is summarized in Figs. 5 and 6. Figure 5 is a plot of the asymptotic mean value $\Phi_{\min}$

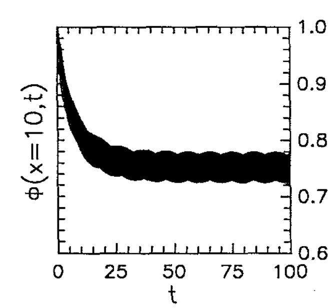

FIG. 2. Time dependence of the kink amplitude for the same set of parameters of Fig. 1. The plot shows $\phi(x = 10, t)$ vs time.

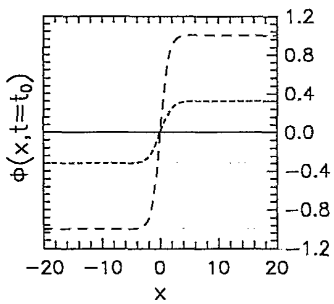

FIG. 3. Same as in Fig. 1 but with $a = 50$. A kink cannot exist under a driving with these parameters.

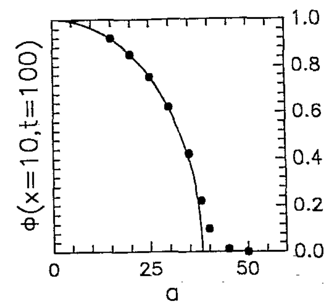

FIG. 5. Kink amplitude $\Phi_{\rm min}$ vs amplitude of the parametrically driving force, a. The plotted numerical values of $\Phi_{\rm min}$ (dots) correspond to $\phi(x = 10, t = 100)$. The solid curve is the analytical prediction.

as obtained from simulations compared to the theoretical prediction Eq. (19) for $\omega = 25$ fixed. $\Phi_{\rm min}$ is computed as explained above and we plot its value at time $t = 100$, in the asymptotic regime. As can be seen from Fig. 5, our analytical approach is actually excellent, although some small discrepancy can be noticed at the large-amplitude part of the plot. This is related to the fact that we have observed that the time needed to reach an asymptotic state, at least as clearly as in Fig. 2, grows with increasing a. Thus, the points for a around 40 or above in Fig. 5 could be closer to the theoretical line, because at $t = 100$ when they were computed, the asymptotic state had not been entered. Much longer runs would be needed, but we have not proceeded with them because they would consume a lot of CPU time, and we feel that the fair agreement between theory and simulations has been already well established. This agreement is further supported by Fig. 6, in which we plot the asymptotic mean value, as well as our theoretical prediction, for $a = 25$ fixed. The accord of theory and simulations is again very good.

IV. CONCLUSIONS

In conclusion, we have analytically and numerically studied the structural stability of kinks in the discrete ϕ^4 model under the action of a periodic parametric driving. By averaging over the fast oscillations, we have shown that in a certain region of the external frequencies it is possible to observe decay of kinks, which would correspond to the case when the averaged, effective potential of the model is transformed from a double-well shape to a single-well one. This prediction has been confirmed by direct numerical simulations, not only qualitatively but also quantitatively, of the parametrically driven ϕ^4 chain with loss, which allow one to settle on a firm basis the phenomenon of kink decay due to this periodic forcing. The obtained results will be useful for analyzing the kink dynamics in the presence of random parametric fluctuations, i.e., multiplicative noises (see, e.g., Ref. [12]), which can be considered as a set of parametric forces with random amplitudes and different frequencies. As has been shown in the present paper, such a parametric force can lead to the disappearance of kinks, so that kink dynamics under the influence of multiplicative noises must exhibit a lot of peculiarities when the noise amplitude is large.

In conclusion, we would like to point out here that the method that we have used in this work may be applied to other parametrically driven nonlinear models like the sine-Gordon one [22]. In this latter case, for instance, a high-frequency parametric force may support the stable propagation of π kinks [16].

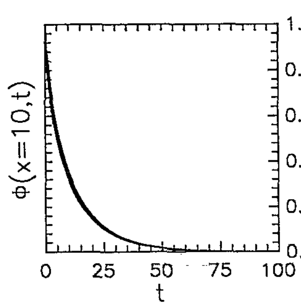

FIG. 4. Same as in Fig. 2 but with $a = 50$. The kink decay is clearly appreciated.

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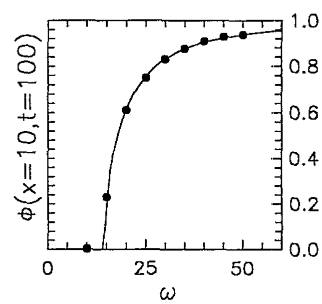

FIG. 6. Kink amplitude $\Phi_{\rm min}$ vs frequency of the parametrically driving force, ω. The plotted numerical values of $\Phi_{\rm min}$ (dots) correspond to $\phi(x = 10,\ t = 100)$. The solid curve is the analytical prediction.

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